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Inverse and Optimal Control for Desired Outputs

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This paper considers the design of control feedforward histories to obtain desired outputs, using inverse and optimal control methods. Some advantages and disadvantages of both methods are presented. The slow roll maneuver of an aircraft is used as an example. A nonlinear inverse solution for this example was given previously; for apparently reasonable specified outputs, namely a roll angle history and a straight flight path, the control inputs and the aircraft sideslip are infeasible and the maneuver is not coordinated (large sideforce). Using optimal control, the mean square control inputs are minimized with only the final roll angle specified. The result is a well-coordinated maneuver with small sideslip, reasonable control amplitudes, and small deviations from the straight flight path.

Introduction

IN the last 15 years, considerable progress has been made in inverse control design (e.g., Refs. 1-7). This paper considers only the determination of control input histories (feedforward controls) to obtain desired outputs. This is only a part of the control design, since feedback control is almost always necessary for a satisfactory implementation. However, feedback is used mainly to provide or augment stability and to handle disturbances and model inaccuracies. The output histories and the control input histories are determined largely by the feedforward logic.

Design Problem 1—Following a Specified Output History

This is the problem considered in this paper; we wish to find the input control history $u(t)$ so that a given system

$$\dot{x} = f(x,u), \qquad x(0) = x_0 \tag{1}$$

$$y = h(x,u) \tag{2}$$

has the output history

$$y(t) = y_m(t) \tag{3}$$

where $y_m(t)$ is specified.

The outputs $y_m(t)$ can be specified by 1) a table of numbers, $y_m(t_i)$, $i = 1, \ldots, N$; 2) an analytic expression in terms of tabulated functions; and 3) the output of an exogenous dynamic model ("exogenous" meaning no forcing functions, only initial conditions). In this case, it is common to choose a linear time-invariant model

$$\dot{x}_m = F_m x_m, \qquad x_m(0) = x_{m0} \tag{4}$$

$$y_m = H_m x_m \tag{5}$$

Design Problem 2—Making the Given System Behave Like Another System

This problem is closely related to problem 1, but the objective is different. Here the designer wishes to find $u = u(x_m, u_m)$ so that the given system, Eqs. (1) and (2), behaves like another dynamic system, e.g., the linear system:

$$\dot{x}_m = F_m x_m + G_m u_m, \qquad x_m(0) = x_{m0} \tag{6}$$

$$y_m = H_m x_m + L_m u_m \tag{7}$$

That is, we want $y(t) = y_m(t)$ for *arbitrary* $u_m(t)$.

If $u_m(t)$ is *specified*, instead of being arbitrary, then this problem reduces to problem 1, since the output is then also specified. The $u_m(t)$ could be specified numerically, analytically, or as the output of still another dynamic system.

Cancellation Compensator Method

An inverse control method was advocated in Ref. 1, which gives a solution to design problem 2 for linear systems. It is a *cancellation compensator* method; the transfer function matrix of the open-loop system is canceled and replaced by a transfer function matrix that produces the desired outputs.

If $u_m(t)$ is specified, the desired transfer function matrix plays a role similar to the output command generator in the previous problem 1.

This method was extended to *nonlinear systems* in robotics where it is called "computed torque"[8] and in aircraft technology (see, for example, Refs. 3, 5, and 7). Here, the nonlinear dynamics are canceled by nonlinear feedback and replaced by linear feedback that produces a desired transfer function matrix.

As pointed out in Ref. 1, this method does not work on unstable systems or systems with "cascade" behavior, i.e., systems where the controls are used to change intermediate outputs that, in turn, affect the desired outputs (right-half plane transmission zeros in the transfer function matrix of linear systems). Canceling a right-half-plane pole with a transmission zero or a right-half-plane transmission zero with a pole creates unstable internal dynamics. It follows that this method also gives poor results for systems with lightly damped left-half-plane poles or zeros.

More generally, cancellation compensators are often nonrobust to plant parameter variations. Also, for multi-input/multi-output (MIMO) systems, such pole assignment methods often produce poorly coordinated (hence large amplitude) controls and awkward maneuvers.

Particular Solution Method

Another inverse control method is to determine a particular solution of the system differential equations that produces the desired outputs. This can be done even for unstable or cascade systems. The particular solution seldom satisfies the desired initial and final conditions for the maneuver. However, prop-

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erly designed feedback will provide or augment the stability of the given system and thus quickly attenuate the initial and final condition errors.[2]

A particular solution method for solving case 3 of problem 1 was given in Ref. 2. The exogenous model was called a *command generator*, and the method was called the *asymptotic model following*. The particular solution is fed forward and the output error is fed back to attenuate initial conditions that are inconsistent with the particular solution and to take care of disturbances and modeling errors.

The particular solutions consist of feedforward gains on the states of the command generator; the gains are found by solving an unsymmetric Lyapunov equation with a linear matrix equation constraint. Later the senior author discovered that the combined feedforward/feedback problem could also be cast as a singular linear-quadratic-follower problem, where the system is augmented with the command generator and penalties are placed only on the output errors (differences between outputs and commanded outputs). The linear quadratic regulator (LQR) solution gives feedback gains on the plant states and on the command generator states (some control software will handle such singular problems, whereas others will not); the latter are really feedforward gains since the output model is uncontrollable. These gains are identical to the combined feedforward and feedback gains found in Ref. 2.

A clever method for finding nonlinear particular solutions was given in Ref. 4. This method is straightforward and much simpler than optimal control methods. However, it turns out that, in some problems, there are no possible control histories for apparently reasonable output histories, or secondary outputs may be unacceptable, or the controls may be poorly coordinated, causing large control amplitudes and awkward maneuvers. The designer can then try to modify the specified output histories until he achieves feasible and coordinated controls, but this may be difficult since it is an indirect procedure.

Optimal Control Methods

Using optimal control methods, the control designer can specify desired output histories or only the desired end results of the maneuver. In addition he has the option to include constraints on the control amplitudes, on initial and final conditions, and on secondary outputs.

Optimal control algorithms quickly show whether or not the desired outputs are possible with feasible controls and feasible secondary outputs. Furthermore, the controls are always well coordinated, producing graceful maneuvers.

Linear-quadratic (LQ) feedback control design methods stabilize unstable systems and produce poles near the *reflected* transmission zeros of cascade systems. This latter fact explains the limitation on possible bandwidth of non-minimum-phase systems; as Ref. 9 states, "Non-minimum-phase systems are fundamentally deficient for regulation."

A disadvantage of nonlinear optimal control algorithms is that they are more complicated to program and use. The LQ algorithms are available in professional software and are very simple to use.

These points are illustrated next by a three-input, three-output example (the slow roll maneuver of an aircraft) that was previously considered[4] using nonlinear inverse control methods.

Example—Slow Roll of an Airplane

In this aerobatic maneuver, the aircraft rolls through 360 deg while staying on an approximately straight horizontal flight path. Assuming negligible change in forward velocity, there are three controls (aileron, elevator, and rudder) and three desired output histories (roll angle and the two components of displacement perpendicular to the desired straight flight path).

Equations of Motion

Nonlinear rigid-body kinematics and linear dynamics were used in Ref. 4. The forward velocity u is assumed constant V, angle of attack α, and the sideslip angle β are assumed small compared with 1 rad. Since the pitch and yaw Euler angles are also small in their solution, it is possible to also linearize the kinematics.

With these assumptions, the dynamic equations, using body principal axis components of velocity (V, $V\beta$, $V\alpha$) and angular velocity (p, q, r) as state variables are

$$\frac{dV}{dt} = 0 \tag{8}$$

$$V(\dot{\alpha} - q + p\beta) = Z_\alpha \alpha + Z_{\delta e}\delta e + g\cos\phi \tag{9}$$

$$V(\dot{\beta} + r - p\alpha) = Y_\beta \beta + Y_{\delta e}\delta r + g\sin\phi \tag{10}$$

$$\dot{p} = L'_\beta \beta + L'_r r + L'_p p + L'_{\delta r}\delta r + L'_{\delta a}\delta a \tag{11}$$

$$\dot{q} + \mu pr = M_\alpha \alpha + M_q q + M_{\delta e}\delta e \tag{12}$$

$$\dot{r} - \sigma pq = N'_\beta \beta + N'_r r + N'_p p + N'_{\delta r}\delta r + N'_{\delta a}\delta a \tag{13}$$

where $\mu = (I_x - I_z)/I_y$, $\sigma = (I_y - I_x)/I_z$, and the nomenclature is that of Ref. 10.

The kinematic equations become

$$\dot{\phi} \cong p \tag{14}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{\psi} \end{bmatrix} \cong \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix} \tag{15}$$

$$\dot{x} \cong V \tag{16}$$

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} \cong V \begin{bmatrix} \psi \\ -\theta \end{bmatrix} + V \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} \tag{17}$$

where (ψ, θ, ϕ) are the NASA standard Euler angles. These equations are linear if $\phi(t)$ is specified, as it is in Ref. 4.

Inverse Control Solution

In Ref. 4, a slow roll maneuver is computed for the A-4D aircraft at 15,000 ft and Mach 0.6 (data from Ref. 10), with the center of mass moving on a straight horizontal path. The method is very clever, but, as the authors point out, the results for this problem are infeasible (sideslip angle almost 20 deg, rudder angle in excess of 30 deg). For the same aircraft at another flight condition (sea level and Mach 0.8, where the dynamic pressure is much higher), the controls and the sideslip angle are feasible, but the sideforce would still be unpleasant for a pilot.

They use angle of attack α, sideslip angle β, and roll angle ϕ as intermediate control variables. They specify $\phi(t)$ and zero components of acceleration of the aircraft center of mass in the body-axis y and z directions, i.e., both sides of Eqs. (9) and (10) must be zero.

Their algorithm for finding the particular solution (which also applies to the nonlinear equations of motion) goes as follows:

1) Find $p(t)$ by differentiating the specified $\phi(t)$.
2) In the force equations (9) and (10), guess $\delta e(t)$ and $\delta r(t)$.
3) Find α and β for the specified specific force histories (the first two terms on the right side of the force equations).
4) Find $\dot{\alpha}$ and $\dot{\beta}$ by differentiation.
5) Find q and r from the force equations.
6) Find $\dot{q}$ and $\dot{r}$ by differentiation.
7) Find δe, δr, and δa from the moment equations (11-13); if they have not changed significantly from the previous iteration, then proceed; otherwise go to step 3.
8) Find the Euler angles by integrating Eq. (15).

9) Find the inertial displacements $y(t)$ and $z(t)$ by integrating Eq. (17).

This algorithm converges very rapidly (in three to five iterations in our experience). *The dynamic part uses only derivatives* and thus avoids integrating the equations of motion. A clever $\phi(t)$ history was chosen in Ref. 4 that starts and ends with zero roll rate and zero roll acceleration (see Fig. 1), so that the initial and final conditions are essentially equilibrium level flight.

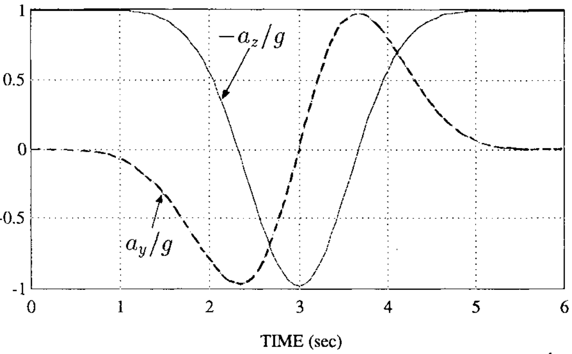

Fig. 1a Specific force histories for slow roll maneuver.[4]

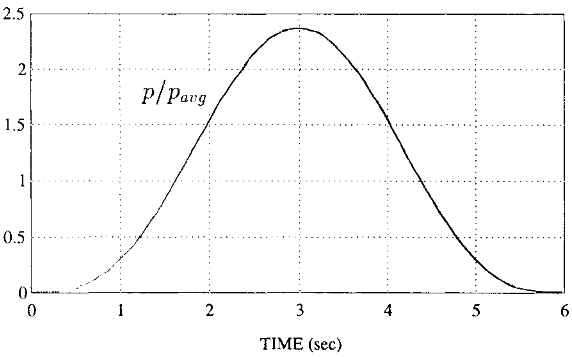

Fig. 1b Specified roll rate history for slow roll maneuver.[4]

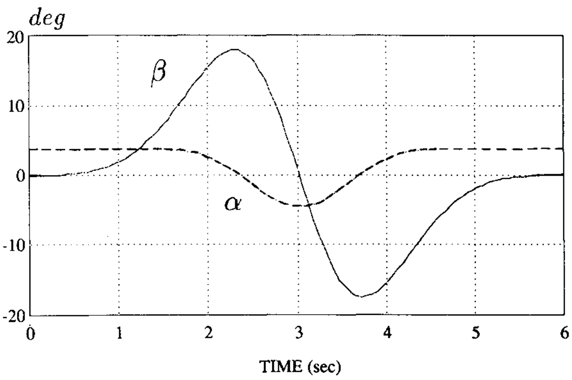

Fig. 2a Angle of attack and sideslip angle histories for slow roll maneuver.[4]

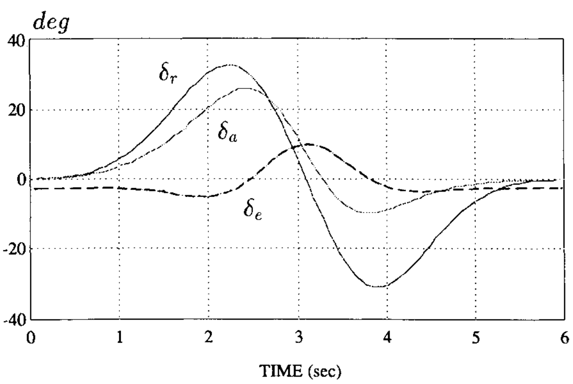

Fig. 2b Control histories for slow roll maneuver.[4]

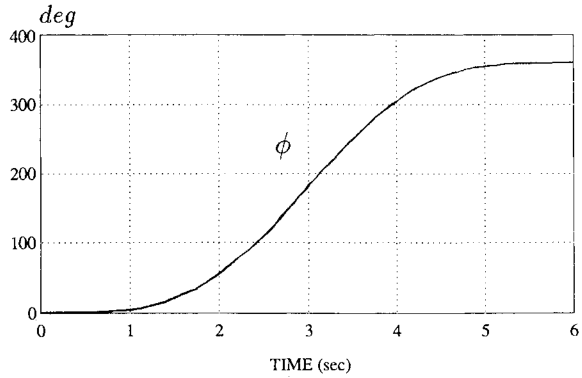

Fig. 3a Roll angle for slow roll maneuver.[4]

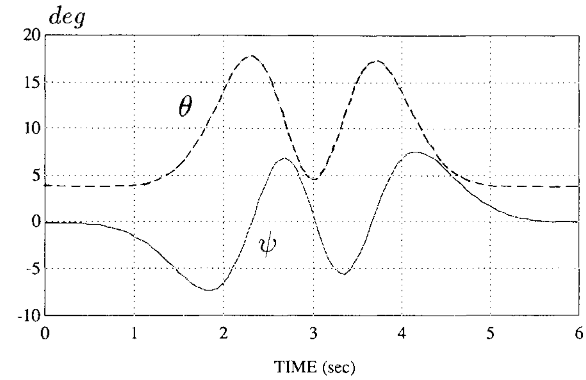

Fig. 3b Pitch and yaw angles for slow roll maneuver.[4]

The unwary might try to find the feedforward controls by using the following feedback method:

1) Using the specified specific force histories and the specified $\phi(t)$, the right-hand sides of Eqs. (9) and (10) are known; this gives δe and δr as linear feedbacks on α and β.

2) Equation (11) gives δa as a feedback on β and r since p and $\dot{p}$ can be obtained by differentiating the specified $\phi(t)$.

3) Integrate this "closed-loop" system forward with the specified $\phi(t)$ as input, to generate the required feedforward control histories.

This method works for stable, noncascade systems. It does not work for this example because it has cascade behavior; for constant roll rate, the system has a right-half-plane transmission zero and the method effectively tries to cancel it with a pole, yielding unstable internal dynamics. However, the feedforward solution exists and can be found by the method of Ref. 4. Note the open-loop system is stable and feedforward does not affect stability.

Figure 2 shows the converged α, β, and control histories from Ref. 4; the sideslip angle β is almost 20 deg, which is infeasible. The rudder deflection exceeds 30 deg, which is also infeasible. Figure 3 shows the Euler angle histories; note θ reaches 18 deg.

If this solution had been feasible, it could have been used as a feedforward in flight by using tabulated control histories and interpolating between times in the table.

Control Constrained Maneuver

The inverse control solution has infeasible controls when specifying the roll angle history and a straight flight path. Hence, using an optimal control algorithm, we minimized the integral-square deviation from a straight flight path with constraints on the controls, specifying only the final conditions, namely roll angle = 360 deg, and final state equal to initial state (horizontal equilibrium flight).

We minimized the performance index

$$J = \int_{t_0}^{t_f} \{y^2 + z^2 + \rho[(\delta e)^2 + (\delta r)^2 + (\delta a)^2]\}\, dt \tag{18}$$

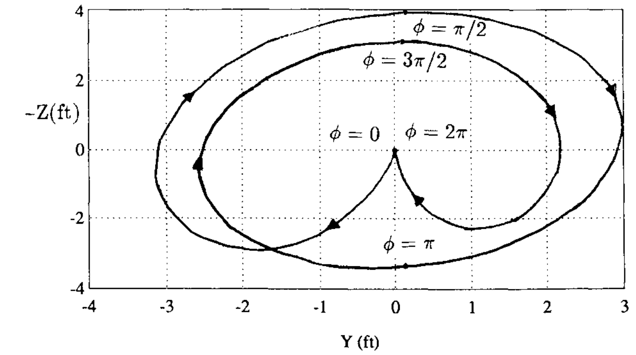

Fig. 4 Inertial displacement of center of mass—slow roll maneuver with minimum control deflections.

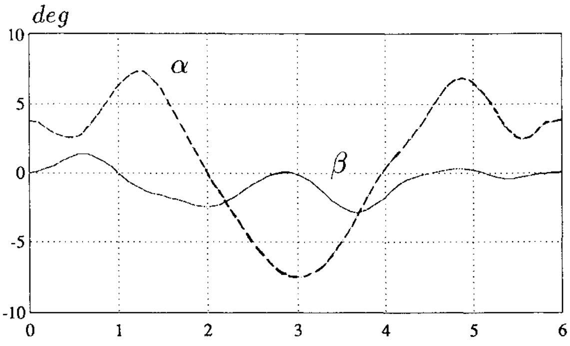

Fig. 5a Angle of attack and sideslip angle histories for slow roll maneuver with minimum control deflections.

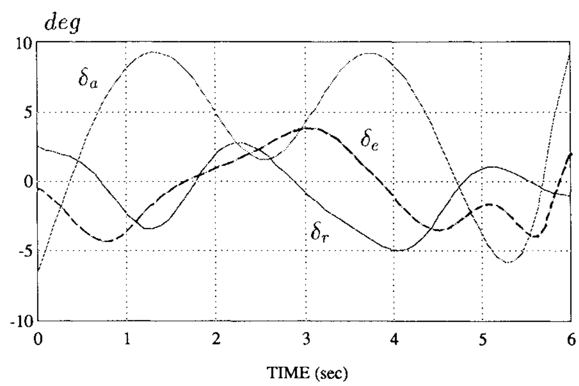

Fig. 5b Control histories for slow roll maneuver with minimum control deflections.

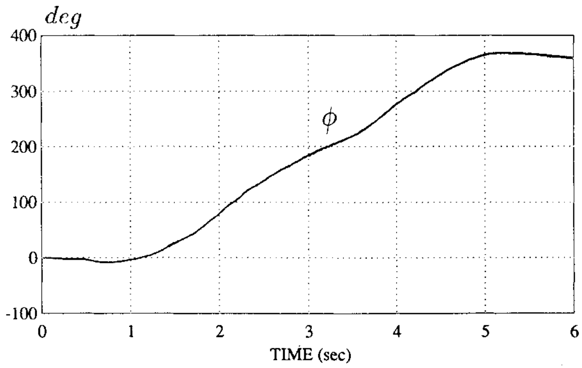

Fig. 6a Roll angle for slow roll maneuver with minimum control deflections.

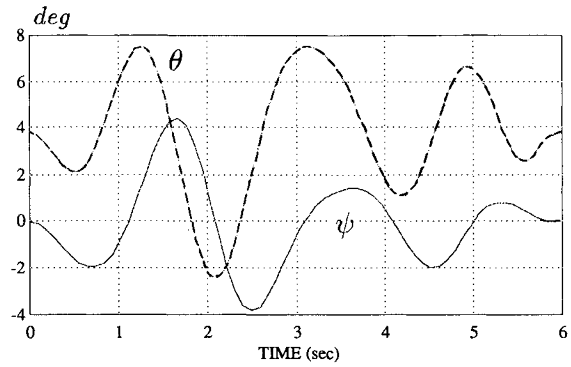

Fig. 6b Pitch and yaw angles for slow roll maneuver with minimum control deflections.

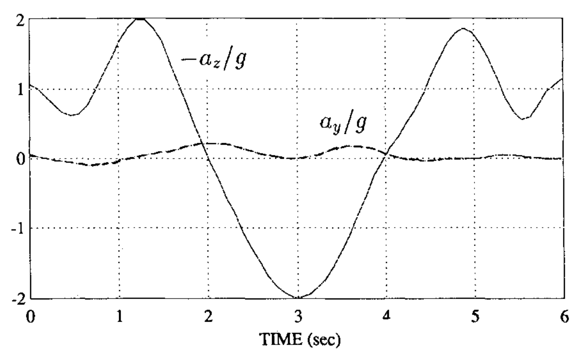

Fig. 7a Specific force histories for slow roll maneuver with minimum control deflections.

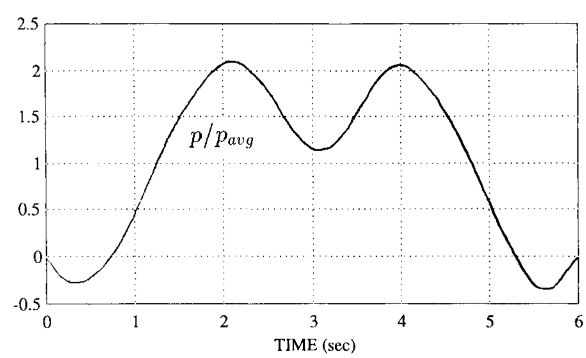

Fig. 7b Roll rate history for slow roll maneuver with minimum control deflections.

The problem was solved with $\rho = 100$ using a gradient projection algorithm[11] with the inverse control solution as the initial guess. The path of the aircraft center of mass is a spiral (Fig. 4), with a maximum displacement from the straight path of about 4.5 ft (the aircraft span is 27 ft). The maximum sideslip angle β is only 3 deg, the maximum aileron angle is 9 deg, and the maximum elevator and rudder angles are both 4 deg (Fig. 5). The Euler angles are shown in Fig. 6; the deviations in both θ and ψ are less than 4 deg.

The control-constrained maneuver is significantly different from the maneuver specified in Ref. 4. The latter maneuver (which cannot be done since it is infeasible) uses sideforce of one g when the aircraft is rolled 90 and 270 deg (Fig. 1). The pilot is subjected to only 1 g at all times, but the direction changes continuously. The minimum control maneuver uses almost no sideforce, but the normal force goes up to 2 g and down to -2 g (Fig. 7). Thus the pilot is subjected to more specific force, but it is almost all in the z direction. Lift is first decreased, then increased, which accelerates the aircraft upward. At the top, the aircraft rolls rapidly and then descends. Negative lift is then increased to stop the aircraft at the bottom and throw it back upward, where it again rolls rapidly.

Conclusions

Inverse control algorithms are relatively straightforward and easy to program. However, for multi-input/multi-output problems, it is sometimes difficult to choose desired output histories that produce feasible, well-coordinated controls and feasible secondary outputs.

Optimal control methods are relatively complicated to program, but if the problem is properly formulated, they produce feasible well-coordinated controls and feasible secondary outputs, with satisfactory output histories.

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